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Wave Growth in M.H.D. Generators

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X. Wave growth in m.h.d. generators

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It has been shown that in an m.h.d. generator, acoustic waves can grow due to the coupling of fluctuations in electrical conductivity, Hall parameter and thermodynamic properties of the gas, with the ohmic dissipation and electromagnetic body forces. A new analysis of this phenomenon is presented in which waves travelling at an arbitrary angle to the flow direction in a plane perpendicular to the magnetic field are considered. In contrast to McCune's (1964) treatment the thermodynamic properties are not restricted to perfect gas laws; and the condition for spatially and temporally growing waves is examined using a general dispersion relation which includes both these types of wave. We consider in detail (i) stationary waves in supersonic flow, and (ii) travelling waves in the subsonic flow found in the C.E.G.B. 200 MW thermal input generator being built at Marchwood, and a possible power station m.h.d. generator. It is found that the waves in the 200 MW rig which burns kerosene in oxygen will be damped. But in an oil-air combustion products generator for Hall parameters of order 3 or greater, it is found that stationary waves which grow rapidly may occur at Mach numbers greater than about 1·7; and in subsonic flow waves propagating antiparallel to the steady current vector may be amplified, though the growth rate is not excessive. In noble gas m.h.d. generators these waves are more unstable than in the oil, air combustion products generator.

1. Introduction

An important question that arises with m.h.d. generators is whether the flow is stable when significant electrical power is extracted from the gas. This problem was first discussed by Velikhov (1962) who showed that an instability could occur in a generator where Hall effects were significant. This instability can be summarized as follows. Consider a local increase in gas pressure which will cause a local decrease in Hall parameter. As shown in figure 1, a Hall current $\mathbf{j}_H$ will be induced, which flows in such a direction as to decrease the angle between the steady current $\mathbf{j}_0$ and steady electric field $\mathbf{E}_0$; it therefore will have

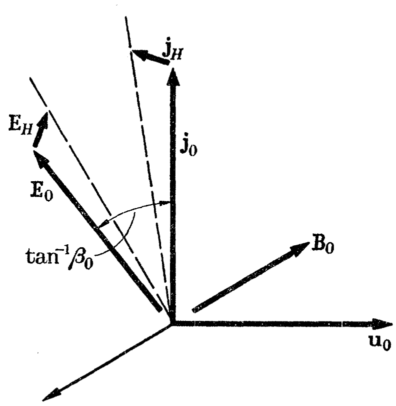

Figure 1. Diagram showing steady current, electric and magnetic field, and velocity vectors illustrating the induced Hall current $\mathbf{j}_H$ of the Velikhov instability.

a component perpendicular to $\mathbf{j}_0$ as shown. This component of $\mathbf{j}_H$ when crossed with the magnetic field $\mathbf{B}_0$, results in a body force which is antiparallel to $\mathbf{j}_0$, and which therefore damps the expansion of the gas parallel to $\mathbf{j}_0$ but amplifies the expansion antiparallel to $\mathbf{j}_0$. Velikhov calculated (with several restrictive assumptions) the condition for this amplifying body force to balance exactly the induced Lorentz body force which acts to damp the disturbance. Wright (1963) considered another possible instability, that of a perturbation in gas conductivity which can be amplified by the resulting change in ohmic dissipation. More recently Sutton & Witalis (1964) considered both these effects together, but restricted their analysis to waves travelling along the duct. McCune (1964) also considered both these effects together and gives an excellent physical description of the phenomena which cause these instabilities. He analysed waves travelling at an arbitrary angle to the flow direction, but he introduced a number of simplifying assumptions including perfect gas equations, and relating the pressure, density and temperature fluctuations by isentropic equations. He analysed only waves which grow or decay with time, and he ignored the possibility of stationary waves in supersonic flow, for which positive growth rates cannot be tolerated at all.

In this paper a more general treatment of the problem is given which removes several of these restrictions, an important extension being that the results can be applied to flows of reacting combustion products. Both spatially and temporally growing waves are considered, and expressions for the growth rates obtained from a general dispersion relation. The results are applied to the predicted flow conditions in the C.E.G.B. 200 MW thermal input m.h.d. generator being constructed at Marchwood which burns kerosene in oxygen, and to a possible power station m.h.d. generator design burning residual oil in air.

2. The dispersion relation

Consider a gas flowing through an m.h.d. generator duct having a pressure, density, temperature, entropy and velocity denoted by p_0, ρ_0, T_0, s_0 and $\mathbf{u}_0$ respectively. The magnetic field strength is $\mathbf{B}_0$, the Hall parameter β_0, the electrical conductivity σ_0, the current density $\mathbf{j}_0$. We wish to examine the amplification with distance and time of a perturbation in each parameter, e.g. pressure. We consider plane waves of the form

$$p = \hat{p}\, e^{i(\omega t - \mathbf{k}.\mathbf{r})}. \tag{1}$$

A coordinate system moving with the gas at velocity $\mathbf{u}_0$ is chosen; and the x axis is alined with the wave vector $\mathbf{k}$, which may be at any angle to the flow direction (see figure 2). The magnetic field is taken to be in the z direction and the unperturbed current $\mathbf{j}_0$ in the x, y plane at an angle θ to the x axis. Only waves travelling in the plane perpendicular to the magnetic field are analysed.

From conservation of charge div $(\mathbf{j}_0+\mathbf{j}) = 0$; and since div $\mathbf{j}_0 = 0$ we obtain div $\mathbf{j} = 0$. Since only plane waves propagating in the x direction are considered, it follows that $\partial j_x/\partial x = 0$; therefore $j_x = 0$ and the perturbed current is transverse to the wave $\mathbf{j} = (0, j)$. Since the magnetic Reynolds number is small, curl $\mathbf{E} = 0$; E_y and E_z are therefore zero and $\mathbf{E}$ is longitudinal $\mathbf{E} = (E, 0)$. Moreover, since $\partial/\partial y$ and $\partial/\partial z$ are zero by hypothesis, $\partial p/\partial y$ and $\partial p/\partial z$ are zero. Also $\mathbf{j}\times\mathbf{B}_0$ has a component only in the x direction, and it follows from the y and z momentum equations that $\partial u_y/\partial t$ and $\partial u_z/\partial t$ are zero. Hence $\mathbf{u}$ is longitudinal, $\mathbf{u} = (u, 0)$.

It is assumed that changes in $\mathbf{u}_0$ are small (i.e. the coordinate system used is an inertial reference frame) which is a reasonable assumption as many generators are designed to operate close to constant velocity. In this coordinate system, therefore, the mean velocity is zero and changes in mean velocity are neglected. Hence for plane waves of the type considered, the linearized continuity, x momentum and energy equations can be written:

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} + u\frac{\partial \rho_0}{\partial x} = 0, \tag{2}$$

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - jB_0 = 0, \tag{3}$$

$$\rho T_0 \frac{\partial s_0}{\partial t} + \rho_0 T \frac{\partial s_0}{\partial t} + \rho_0 T_0 \frac{\partial s}{\partial t} + \rho_0 T_0 u \frac{\partial s_0}{\partial x} = \frac{2 \sin\theta j_0 j}{\sigma_0} - \frac{j_0^2 \sigma}{\sigma_0^2}. \tag{4}$$

In equation (2) the orders of magnitude of the terms are

$$\frac{\partial \rho}{\partial t} \sim \rho_0 \frac{a_0}{\lambda}\frac{\rho}{\rho_0}; \quad \rho_0 \frac{\partial u}{\partial x} \sim \rho_0 \frac{u_0}{\lambda}\frac{u}{u_0}; \quad u\frac{\partial \rho_0}{\partial x} \sim \rho_0 \frac{u_0}{L}\frac{u}{u_0};$$

where L is the length of the generator, λ is the wavelength of the wave, and a_0 is the sound speed. We assume that $L \gg \lambda$, and therefore the term $u\partial\rho_0/\partial x$ can be neglected compared with the term $\rho_0\, \partial u/\partial x$.

In equation (4) the orders of magnitude of the terms on the left are:

$$\rho T_0 \frac{\partial s_0}{\partial t} \sim \frac{\rho_0 T_0 u_0 \Delta s_0}{L}\frac{\rho}{\rho_0}; \quad \rho_0 T \frac{\partial s_0}{\partial t} \sim \frac{\rho_0 T_0 u_0 \Delta s_0}{L}\frac{T}{T_0};$$

$$\rho_0 T_0 \frac{\partial s}{\partial t} \sim \frac{\rho_0 T_0 \Delta s_0 a_0}{\lambda}\frac{s}{\Delta s_0}; \quad \rho_0 T_0 u \frac{\partial s_0}{\partial x} \sim \frac{\rho_0 T_0 u_0 \Delta s_0}{L}\frac{u}{u_0},$$

where Δs_0 is the change in entropy across the generator. For order of magnitude estimates we can treat the gas as a perfect gas, and can write

$$s = C_v T/T_0 - \boldsymbol{R}\, \rho/\rho_0,$$
$$\Delta s_0 = C_v \ln (T_2/T_1) - \boldsymbol{R} \ln (\rho_2/\rho_1),$$

where C_v is the specific heat at constant volume and $\boldsymbol{R}$ is the gas constant. Since the density ratio across the generator, ρ_2/ρ_1, is of order 0·2, and temperature ratio T_2/T_1 of order 0·9, Δs_0 is of order $\boldsymbol{R}$. Thus $s/\Delta s_0$ is of order ρ/ρ_0, or at least of order T/T_0 since $C_v/\boldsymbol{R} > 1$. From the continuity equation u/u_0 is of order ρ/ρ_0 and hence we only retain the term $\rho_0 T_0 \partial s/\partial t$ on the left of equation (4). It can also be shown, from the mean flow energy equation, that the terms on the right of equation (4) are of order:

$$\frac{j_0 j}{\sigma_0} \sim \frac{\rho_0 T_0 \Delta s_0 u_0}{L}\frac{j}{j_0}; \quad \frac{j_0^2 \sigma}{\sigma_0} \sim \frac{\rho_0 T_0 \Delta s_0 u_0}{L}\frac{\sigma}{\sigma_0},$$

and therefore j/j_0 and σ/σ_0 must be greater than ρ/ρ_0, T/T_0 and u/u_0 for these terms to be retained. Equations (8) and (10), which follow, show that this is the case, as the conductivity varies approximately as T^{10} for combustion gases in the range of interest.

The conservation equations can therefore be written as

$$\frac{\partial \rho}{\partial t}+\rho_0\frac{\partial u}{\partial x} = 0, \tag{5}$$

$$\rho_0\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}-jB_0 = 0, \tag{6}$$

$$\rho_0 T_0\left(\frac{\partial s_0}{\partial T_0}\right)_{\rho_0}\frac{\partial T}{\partial t}+\rho_0 T_0\left(\frac{\partial s_0}{\partial \rho_0}\right)_{T_0}\frac{\partial \rho}{\partial t} = \frac{2\sin\theta\, j_0 j}{\sigma_0}-\frac{j_0^2\sigma}{\sigma_0^2}, \tag{7}$$

where in equation (7), $\partial s/\partial t$ has been expressed in terms of $\partial T/\partial t$ and $\partial \rho/\partial t$. Thermal conduction has been omitted from equation (4) as it can be shown that its contribution is negligible for frequencies below 1 Mc/s, provided that the wave growths are not excessive.

Assuming that the electrons are in thermal equilibrium at the gas temperature, the change in electrical conductivity can be determined from Saha's equation and mobility. The linearized equation is

$$\frac{\sigma}{\sigma_0} = -\frac{1}{2}\frac{\rho}{\rho_0}+R\frac{T}{T_0}, \tag{8}$$

where $R = eV_i/(2kT_0)+\frac{1}{4}$; and V_i is the ionization potential of the seed, e is the electronic charge and k is Boltzmann's constant. The linear variation of Hall parameter is given by

$$\frac{\beta}{\beta_0} = -\frac{1}{2}\frac{T}{T_0}-\frac{\rho}{\rho_0}. \tag{9}$$

The perturbation current obtained from the y component of the generalized Ohm law is given by

$$\frac{j}{j_0} = \frac{\sigma}{\sigma_0}(\sin\theta-\beta_0\cos\theta)-\frac{uB_0\sigma_0}{j_0}+\beta\cos\theta. \tag{10}$$

Finally there is the linearized equation of state

$$T = \left(\frac{\partial T_0}{\partial p_0}\right)_{\rho_0} p+\left(\frac{\partial T_0}{\partial \rho_0}\right)_{p_0}\rho. \tag{11}$$

Substitution of the wave equations (1) into equations (5) to (11) with $\partial/\partial t = \mathrm{i}\omega$ and $\partial/\partial x = -\mathrm{i}k$, yields a set of linear equations in the perturbation amplitudes $\hat{\rho}$, $\hat{u}$, $\hat{p}$, $\hat{T}$, $\hat{\beta}$, $\hat{\sigma}$ and $\hat{\jmath}$. The condition of solubility of this set, the dispersion relation, can be arranged in the following form:

$$\left(\frac{a_0 k}{\omega}\right)^2\left\{1-\frac{\mathrm{i}j_0^2}{2\rho_0 C_p T_0\sigma_0\omega}\left[(2R+\mu)\cos 2\theta+\beta_0(2R+1-\mu)\sin 2\theta\right]\right\}$$
$$+\left(\frac{a_0 k}{\omega}\right)\left[\frac{j_0^3 B_0\beta_0}{4a_0\rho_0^2 C_v T_0\sigma_0\omega^2}(4R+1)\cos\theta-\frac{\mathrm{i}j_0 B_0}{2\mu a_0\rho_0\omega}(K_1\sin\theta-K_2\cos\theta)+\frac{\mathrm{i}\,2a_0\mu j_0 B_0}{\rho_0 C_p T_0\omega}\sin\theta\right]$$
$$-1+\frac{j_0^2 B_0^2 R}{\rho_0^2 C_v T_0\omega^2}+\frac{\mathrm{i}\,j_0^2}{\rho_0 C_v T_0\sigma_0\omega}\left[R\cos 2\theta+\beta_0(R+\tfrac{1}{2})\sin 2\theta\right]+\frac{\mathrm{i}\,\sigma_0 B_0^2}{\rho_0\omega} = 0, \tag{12}$$

where

$$\left.\begin{aligned} K_1 &= 2R(C_p/C_v-1)-\mu,\\ K_2 &= \beta_0[(2R+1)(C_p/C_v-1)+\mu]\end{aligned}\right\}. \tag{13}$$

Use has been made of the relations

$$T_0\left(\frac{\partial s_0}{\partial T_0}\right)_{\rho_0}+T_0\left(\frac{\partial s_0}{\partial \rho_0}\right)_{T_0}\left(\frac{\partial \rho_0}{\partial T_0}\right)_{p_0}=T_0\left(\frac{\partial s_0}{\partial T_0}\right)_{p_0}=C_p,$$

$$T_0\left(\frac{\partial s_0}{\partial T_0}\right)_{\rho_0}=C_v,$$

$$a_0^2=\frac{C_p}{C_v}\left(\frac{\partial p_0}{\partial \rho_0}\right)_{T_0},\quad \mu=-\frac{T_0}{\rho_0}\left(\frac{\partial \rho_0}{\partial T_0}\right)_{p_0},$$

where C_p and C_v are specific heats at constant pressure and volume respectively, and a_0 is the sound speed. Equation (12) indicates that the gas velocity u_0 does not enter into the dispersion relation and therefore that the way in which the electric field and current are produced (i.e. induced or applied) does not affect these waves. It also indicates that where $j_0=0$, the only amplifying or damping term in equation (12) which is not zero is the $\mathrm{i}\sigma_0 B_0^2/(\rho_0\omega)$ term which can be shown to represent the damping from the induced Lorentz body forces. Hence instabilities can only occur when electrical power is extracted from or put into the flow.

A characteristic time τ where

$$\tau=\frac{1-\eta_e}{\eta_e}\,\frac{\sigma_0\rho_0 C_p T_0}{j_0^2}$$

is now introduced, where $\eta_e=\mathbf{E}_0\cdot\mathbf{j}_0/(\mathbf{u}_0\cdot\mathbf{j}_0\times\mathbf{B}_0)$ is the electrical efficiency of the generator; τ is the time taken to extract an amount of energy $C_p T_0$ from the gas by the m.h.d. process. If the angle between $\mathbf{u}_0$ and $\mathbf{j}_0$ is ϕ then

$$j_0 B_0=\frac{j_0^2}{(1-\eta_e)\ \sigma_0 a_0 M_0 \sin\phi},$$

where M_0 is the Mach number of the flow. The dispersion relation, equation (12), can now be written as

$$\begin{aligned}&\left(\frac{a_0 k}{\omega}\right)^2\left\{1-\mathrm{i}\,\frac{1-\eta_e}{2\eta_e(\omega\tau)}\left[(2R+\mu)\cos 2\theta+\beta_0(2R+1-\mu)\sin 2\theta\right]\right\}\\&+\left(\frac{a_0 k}{\omega}\right)\left\{\frac{1-\eta_e}{4\eta_e^2}\,\frac{C_p^2 T_0\beta_0}{a_0^2 C_v M_0}\,\frac{(4R+1)\cos\theta}{\sin\phi(\omega\tau)^2}-\frac{\mathrm{i}\,C_p T_0}{2a_0^2\eta_e\mu M_0\sin\phi\ (\omega\tau)}\right.\\&\left.\times[K_1\sin\theta-K_2\cos\theta]+\frac{2\mathrm{i}\ \mu\sin\theta}{\eta_e M_0\sin\phi(\omega\tau)}\right\}-1\\&+\frac{C_p^2 T_0 R}{C_v a_0^2\eta_e^2 M_0^2\sin^2\phi(\omega\tau)^2}+\frac{\mathrm{i}\,C_p(1-\eta_e)}{C_v\eta_e(\omega\tau)}\\&\times[R\cos 2\theta+\beta_0(R+\tfrac{1}{2})\sin 2\theta]+\frac{\mathrm{i}\,C_p T_0}{a_0^2 M_0^2\sin^2\phi(1-\eta_e)\ \eta_e(\omega\tau)}=0.\end{aligned}\tag{14}$$

3. Calculation of the growth rates

It has been assumed that the wavelength of the perturbation is much less than the length of the generator. This is equivalent to the condition that the period of the wave is much less than τ, i.e. $|\omega\tau|\gg 1$. Equation (14) can now be solved neglecting terms of order $(\omega\tau)^{-2}$ to give

$$\left(\frac{a_0 k}{\omega}\right)=1-\frac{\mathrm{i}}{\omega\tau}f(\theta),\tag{15}$$

where $f(\theta)$ is given by

$$f(\theta) = \frac{C_p T_0}{4a_0^2 \eta_e \mu M_0 \sin\phi}\left[K_2 \cos\theta - K_1 \sin\theta + \frac{2\mu}{M_0 \sin\phi(1-\eta_e)}\right] + \frac{\mu \sin\theta}{\eta_e M_0 \sin\phi} + \frac{1-\eta_e}{4\eta_e}[K_1 \cos 2\theta + K_2 \sin 2\theta], \qquad (16)$$

with K_1 and K_2 defined by equation (13).

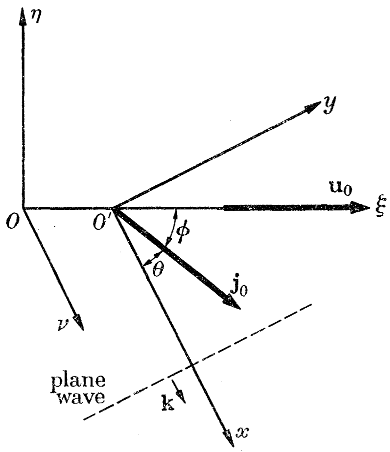

FIGURE 2. Relation between moving (x, y, z) and stationary (ξ, η, ζ) coordinate systems.

We now transform from the moving coordinates system (x, y, z) to a coordinate system at rest (ξ, η, ζ) where ξ is alined parallel to $\mathbf{u}_0$ and ζ parallel to $\mathbf{B}_0$. Figure 2 illustrates the geometry. The relation between the coordinates x, ξ and η can be written as

$$x = \xi \cos(\theta+\phi) - \eta \sin(\theta+\phi) - u_0 t \cos(\theta+\phi). \qquad (17)$$

The plane wave equation (1) can be expressed as

$$p = \hat{p} \exp[-\omega_i t + k_i x + \mathrm{i}(\omega_r t - k_r x)], \qquad (18)$$

where subscripts r and i denote the real and imaginary parts of ω and k. In the moving x, y, z coordinate system the disturbance grows spatially when $k_i > 0$,* and grows temporally when $\omega_i < 0$. For the stationary coordinate system these criteria must be changed. On substitution of equation (17) into equation (18), the wave equation becomes

$$p = \hat{p} \exp(-[\omega_i + k_i u_0 \cos(\theta+\phi)]\, t + k_i[\xi \cos(\theta+\phi) - \eta \sin(\theta+\phi)] + \mathrm{i}\{[\omega_r + k_r u_0 \cos(\theta+\phi)]\, t - k_r[\xi \cos(\theta+\phi) - \eta \sin(\theta+\phi]\}) \qquad (19)$$

and the disturbance grows spatially if $k_i > 0$, and temporally if $[\omega_i + k_i u_0 \cos(\theta+\phi)] < 0$.

Equation (15) can be expressed in real and imaginary parts as

$$k_r a_0 = \omega_r; \quad k_i a_0 = \omega_i - f(\theta)/\tau, \qquad (20)$$

and it follows that, in the ξ, η, ζ coordinate frame,

(i) for a wave whose growth is time dependent only (i.e. $k_i = 0$), the wave equation (19) becomes

$$p = \hat{p} \exp\{-f(\theta)t/\tau\} \exp\{\mathrm{i}\, \omega_r [1 + M_0 \cos(\theta+\phi)]t - \mathrm{i}\, k_r \nu\}, \qquad (21a)$$

* With $0 \leqslant \theta \leqslant 2\pi$ only positive x is considered.

where $\nu = \xi \cos(\theta+\phi) - \eta \sin(\theta+\phi)$ is distance measured from the fixed origin 0 parallel to $\mathbf{k}$ (see figure 2). The wave amplifies if $\omega_i = f(\theta)/\tau < 0$ with a growth rate proportional to $\exp[-f(\theta)t/\tau]$;

(ii) for a wave whose growth is space dependent only [i.e. $\omega_i + k_i u_0 \cos(\theta+\phi) = 0$], equation (19) becomes

$$p = \hat{p} \exp\left\{\frac{-f(\theta)\nu}{a_0\tau[1+M_0\cos(\theta+\phi)]}\right\} \exp\{\mathrm{i}\omega_r[1+M_0\cos(\theta+\phi)]t - \mathrm{i}\,k_r\nu\}, \qquad (21\,b)$$

and the wave amplifies if $k_i = -f(\theta)/\{a_0\tau[1+M_0\cos(\theta+\phi)]\} > 0$ with a growth rate proportional to $\exp(-f(\theta)\nu/\{a_0\tau[1+M_0\cos(\theta+\phi)]\})$. These growth rates for two cases of specific interest will now be examined.

4. Stationary waves in supersonic flow

In a supersonic flow waves which are stationary with respect to the duct can exist when the apparent axial wave speed is equal and opposite to $\mathbf{u}_0$. Equations (21 a, b) show that the two critical angles are given by

$$1+M_0\cos(\theta+\phi) = 0. \qquad (22)$$

Equation (21 b) shows that for this special case, the spatial rate of amplification of these waves is infinite. However, this singularity occurs because terms of order $(\omega\tau)^{-2}$ were omitted from equation (15), and when the second order terms are included we find that $(k_i a_0 \tau)$ is of order $(\omega_r\tau) \gg 1$, as compared with a value of order 1 predicted by equation (21 b) at angles other than these critical angles. Additional effects to consider are heat conduction and viscosity which will become important as the rate of amplification of the waves increases. We have ignored these effects, since by including them the order of the dispersion relation is increased, and for slowly growing waves these higher order terms are negligible. It is only for $k_i \gg 1$ that these terms become of the same order as the terms retained, and thus it is only in this limiting case that viscosity and heat conduction can contribute. Clearly these stationary waves are potentially the most unstable, and it is of considerable practical importance to calculate the range of M_0 (for $M_0 > 1$) in which the waves amplify, i.e. $f(\theta)$ is negative. We use the expression for the critical angle, equation (22), to eliminate θ from equation (16). The following expression for $f_{\text{crit.}}$ results:

$$\begin{aligned} f_{\text{crit.}} = {} & \frac{1-\eta_e}{2\eta_e M_0^2}(K_1\cos 2\phi - K_2 \sin 2\phi) + \frac{\mu}{\eta_e M_0^2} \\ & + \frac{C_p T_0}{4a_0^2 M_0^2\,\eta_e\,\mu \sin\phi}\left[\frac{2\mu}{(1-\eta_e)\sin\phi} - K_2\cos\phi - K_1\sin\phi\right] \\ & \pm \frac{(M_0^2-1)^{\frac{1}{2}}}{M_0^2}\left[\frac{\mu}{\eta_e}\cot\phi - \frac{1-\eta_e}{2\eta_e}(K_1\sin 2\phi + K_2\cos 2\phi)\right. \\ & \left. + \frac{C_p T_0}{4\eta_e a_0^2\,\mu\sin\phi}(K_2\sin\phi - K_1\cos\phi)\right] + \frac{1-\eta_e}{4\eta_e}(K_2\sin 2\phi - K_1\cos 2\phi). \end{aligned} \qquad (23)$$

The plus sign corresponds to waves where $\mathbf{k}$ has a component parallel to $\mathbf{j}_0$ (i.e. angle between $\mathbf{k}$ and $\mathbf{j}_0$ is less than $\frac{1}{2}\pi$); the minus sign where $\mathbf{k}$ has a component antiparallel

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to $\mathbf{j}_0$. Only the case $\phi = \frac{1}{2}\pi$ will be treated in detail, since in a well designed generator the current flows nearly normal to the gas flow direction. For this value of ϕ equation (23) can be simplified to show that $f_{\text{crit.}}$ will be negative and the disturbance amplifies when

$$K_3[2\mu - K_1(1-\eta_e) \pm K_2(1-\eta_e)(M_0^2-1)^{\frac{1}{2}}] + (1-\eta_e)^2 K_1 M_0^2 < 0, \tag{24}$$

where
$$K_3 = 2(1-\eta_e) + C_p T_0/(a_0^2 \mu)$$

and the plus and minus signs are taken as explained above.

For kerosene–oxygen combustion product generators $\mu \approx 2$, $C_p/C_v \approx 1{\cdot}06$, $R \approx 10$, $\eta_e \approx 0{\cdot}7$, $\beta_0 \approx 1$, $C_p T_0/a_0^2 \approx 25$; which gives $K_1 = -1{\cdot}0$, $K_2 = 3{\cdot}0$, $K_3 = 13$ and the inequality (24) indicates that the wave with $\mathbf{k}$ having a component antiparallel to $\mathbf{j}_0$ is unstable for $M_0 > 4{\cdot}5$ which is outside the range of interest. The other wave is always stable.

For residual oil-air combustion products generators with $\mu \approx 1{\cdot}1$, $C_p/C_v \approx 1{\cdot}13$, $C_p T_0/a_0^2 \approx 7$, $R \approx 11$, $\beta_0 \approx 2$, $\eta_e \approx 0{\cdot}7$; which gives $K_1 = 1{\cdot}8$, $K_2 = 4{\cdot}1$, $K_3 = 7{\cdot}0$ and inequality (24) indicates this same antiparallel wave unstable for $M_0 > 1{\cdot}7$ which may not be outside the range of interest. Increasing the Hall parameter (increasing K_2) decreases this critical Mach number, and for an air-oil combustion products, Hall generator with $\beta_0 \approx 8$, $\eta_e \approx 0{\cdot}6$ and other properties as above, the antiparallel stationary waves are unstable for $M_0 > 1{\cdot}02$. In a Hall generator, the currents will have a component along the flow direction, and ϕ is somewhat less than $\frac{1}{2}\pi$. It can be shown by expanding equation (23) in a power series in $(\frac{1}{2}\pi - \phi)$ that this tends to destabilize the generator.

In an inert gas generator $\mu \approx 1$, $C_p/C_v = \frac{5}{3}$, $R \approx 13$, $\eta_e \approx 0{\cdot}7$, $\beta_0 \approx 2$, $C_p T_0/a_0^2 \approx 1{\cdot}5$ giving $K_1 = 16$, $K_2 = 37$ and $K_3 = 2{\cdot}1$ inequality (24) shows that antiparallel stationary waves are always unstable for $M_0 > 1$. Note, however, that in all these cases the possible damping effect of the duct walls has not been considered.

5. Subsonic flow in combustion products generators

The design conditions at duct entrance (*a*), and exit (*b*), for the C.E.G.B. 200 MW rig with kerosene–oxygen combustion products; and approximate inlet (*c*), and outlet (*d*), conditions for an oil-air combustion products generator for a power station are given in table 1. The attenuation or amplification with distance of acoustic waves at an arbitrary angle $(\theta+\phi)$ to the flow direction can now be calculated from equations (16) and (21*b*). The value of $\alpha = f(\theta)/\{a_0\tau[1+M_0\cos(\theta+\phi)]\}$ (unit m^{-1}) as a function of $(\theta+\phi)$ is shown in figure 3A for cases (*a*) and (*b*), and figure 3B for cases (*c*) and (*d*). The wave attenuates for positive α, amplifies for negative α, as $e^{-\alpha\nu}$ where ν is distance (in m) parallel to the wave vector $\mathbf{k}$. Cases (*b*), (*c*) and (*d*) show the effect of the Hall current body force first discussed by Velikhov (1962) which acts to amplify waves antiparallel to $\mathbf{j}_0$ and damp waves parallel to $\mathbf{j}_0$. In cases (*b*) and (*c*) the antiparallel wave $(\theta+\phi \approx 270°)$ is less damped than the parallel wave $(\theta+\phi \approx 90°)$, and in case (*d*) the antiparallel wave is unstable though the growth rate is not excessive. Case (*a*) does not show this effect since for these conditions with $\beta_0 \approx 0{\cdot}3$ Hall effects are small.

For inert gas generator conditions it can be shown from equations (16) and (21*b*) that waves antiparallel to $\mathbf{j}_0$ are more unstable than the combustion products case (*d*) considered above, and the growth rates are larger.

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We conclude that in the C.E.G.B. 200 MW rig disturbances of the type considered in this paper are damped. However, in oil–air combustion products generators, waves anti-parallel to the current $\mathbf{j}_0$ may be amplified slowly at the low pressure end of the generator. Again the effect of duct walls has not been considered.

Table 1. Values of duct entrance and exit parameters for 200 MW rig (kerosene–oxygen combustion products) and for a possible power station design (residual oil-air combustion products).

	200 MW rig, kerosene–oxygen		power station, residual oil–air	
	entrance	exit	entrance	exit
C_p/C_v	1·06	1·07	1·13	1·12
$C_p T_0/a_0^2$	24	27	8·2	5·4
μ	2·0	1·9	1·1	1·1
R	8·2	9·2	10·3	12·3
β_0	0·33	2·0	1·1	3·5
M_0	0·8	0·8	0·9	0·9
$a_0\tau$(m)	430	90	96	114
η_e	0·34	0·66	0·7	0·7
ϕ (deg)	104	83	90	90

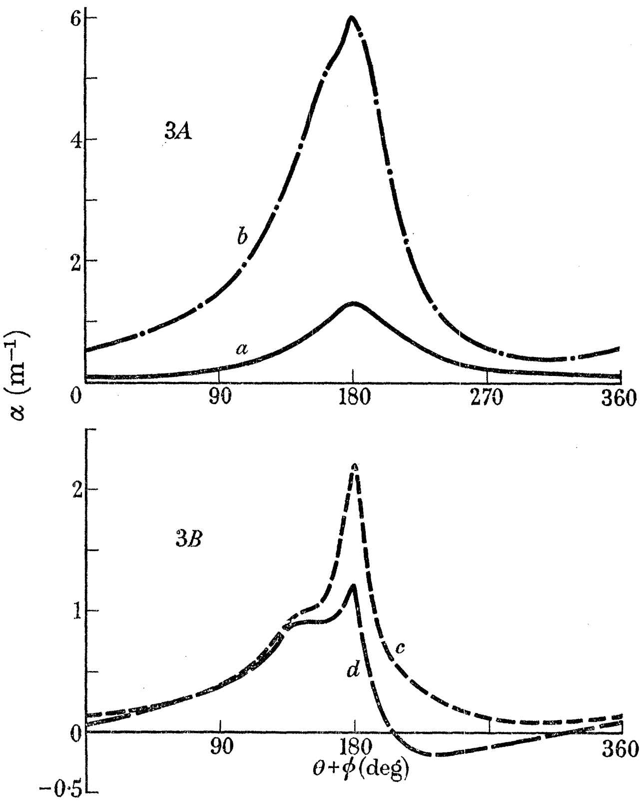

Figure 3. Curves showing attenuation of acoustic waves as a function of angle $(\theta+\phi)$ between the wave vector and flow direction. Waves attenuate as $e^{-\alpha\nu}$ where ν is distance (metres) parallel to **k**. Figure 3A shows cases (*a*) and (*b*) C.E.G.B. 200 MW rig (kerosene–oxygen products) duct entrance and exit conditions respectively. Figure 3B shows cases (*c*) and (*d*) possible power station generator (residual oil–air combustion products) duct entrance and exit conditions respectively. Note the different vertical scales.

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